APPENDIX B

The primitive equations

The physical and mathematical basis of all methods of dynamical atmospheric prediction lies in the principles of *conservation of momentum, mass, and energy*. Applied to the quasi-continuous statistical motion of an assemblage of liquid or gas molecules (through the methods of kinetic theory and statistical mechanics), these fundamental principles are expressed mathematically to form a set of equations called primitive equations. The primitive equations are expressed either in geometrically fixed coordinates or in a hybrid system of coordinates. Although it is easiest to give physical interpretation to the primitive equations in geometrically fixed coordinates, it is simpler from the mathematical standpoint and yet more practical to refer the primitive equations to a hybrid system of coordinates, in which two sets of coordinate surfaces are geometrically fixed and the vertical coordinate is moving. For example, the two famous and widely used vertical coordinates in expressing the primitive equations are "pressure coordinates" developed by Eliassen, and "theta coordinates" developed by Montgomery and Starr (Thompson, 1961).

A special type of vertical coordinate involved with the hybrid system is the "sigma" coordinate. This σ-coordinate system, originally proposed by Phillips (1957) is the most commonly used in modelling because of its advantages. The use of this σ-coordinate system at the bottom levels helps in the application of realistic boundary conditions (Cullen, 1993). The lower boundary conditions require that this coordinate be used at the lower boundary in most models. The Unified Model also uses hybrid vertical coordinates, which can be specified as pressure or sigma or a combination of the two in its vertical levels. The form of the primitive equations here, therefore, will be illustrated in this coordinate system. In a mathematical form, this σ-coordinate is simply given by:

$$\sigma = p / p_s \tag{B.1}$$

where p is pressure and p_s is surface pressure. With this system of coordinates, a set of primitive equations can be formulated according to each of the fundamental considerations, which can be briefly described as follows:

(i) The Newtonian (or Navier-Stokes) equations of motion, relating the acceleration in any given direction to the component of force per unit mass acting in that direction for a continuous medium.

(ii) The thermodynamic energy equation, stating that any heat energy added to a system must be equal to the change in its internal energy plus the work done by it in expanding against the pressure force.

(iii) The equation of continuity (for mass conservation), which simply states that any local increase or decrease in mass density (and moisture) must be exactly balanced by a corresponding import or export of mass (and moisture) in that region.

(iv) The equation of hydrostatic equilibrium, simply states that the gravitational acceleration g is almost exactly balanced by the acceleration due to the atmosphere's buoyancy.

Mathematically expressed with the inclusion of parameterization terms, following Simmons and Bengtsson (1984), the equations are listed as follows: -

i. Conservation of momentum

$$\frac{\mathrm{D}\mathbf{v}}{\mathrm{D}t} + f\mathbf{k} \times \mathbf{v} + \nabla\phi + R_d T \,\nabla \ln p_s = \mathbf{P}_v + \mathbf{K}_v \tag{B.2}$$

ii. Conservation of energy

$$\frac{\mathrm{D}T}{\mathrm{D}t} - \frac{R_d T \omega}{c_{pd}\, p_s \sigma} = P_T + K_T \tag{B.3}$$

iii. (a) Conservation of moisture

$$\frac{\mathrm{D}q}{\mathrm{D}t} = P_q + K_q \tag{B.4}$$

(b) Conservation of mass

$$\frac{\mathrm{D}p_s}{\mathrm{D}t} + p_s\left(\nabla \cdot \mathbf{v} + \frac{\partial \dot{\sigma}}{\partial \sigma}\right) = 0. \tag{B.5}$$

iv. Hydrostatic equilibrium

$$\frac{\partial \phi}{\partial \sigma} = -\frac{R_d T}{\sigma} \tag{B.6}$$

For the above set of primitive equations, t is time and D/Dt denotes the rate of change moving with a fluid particle, which in σ-coordinates takes the form:

$$\frac{\mathrm{D}}{\mathrm{D}t} = \frac{\partial}{\partial t} + \mathbf{v} \cdot \nabla + \dot{\sigma} \frac{\partial}{\partial \sigma};$$

where **v** is the horizontal velocity vector, $\mathbf{v} = (u, v, 0)$, and ∇ is the two-dimensional gradient operator on a surface of constant σ; ω is pressure vertical velocity; f is the Coriolis parameter (twice the earth's angular rotation rate multiplied by the sine of latitude); **k** is the unit vertical vector; ϕ is the geopotential (the acceleration due to gravity multiplied by the height of the surface of constant pressure); R_d is the gas constant for dry air, and C_{pd} the specific heat of dry air at constant pressure.

The term P_x in the equations for conservation of momentum, energy and moisture denotes the rate of change of variable X due to the parameterized processes of radiation, convection, turbulent vertical mixing, large-scale precipitation and others. Within the same equations, the term K_x represents the rate of change of X due to the explicit horizontal smoothing that is usually included in models to prevent an unrealistic growth of the smallest resolved scales. This term would ideally be regarded as representing the influence of unresolved scales of motion on the explicitly predicted scales, also treated as part of the parameterization. In practice, since the smallest scales in a model are inevitably subject to numerical misrepresentation, it is common to choose empirically a computationally convenient form of smoothing, and to adjust it so that contour plots of the predicted variables do not appear excessively rough (Simmons and Bengtsson, 1984).

For the purpose of numerical calculation in a model, the vector equation (B.2) must be decomposed into three scalar equations, corresponding to its components in three different coordinate directions.

A number of approximations or assumptions have been made in solving the system of equations. The thermodynamic energy equation for conservation of energy can be integrated by making use of the Boyle-Charles equation of state, $p = \rho RT$. With the consideration of equation (B.6), then the model atmosphere is always in hydrostatic balance, with no vertical accelerations. The set of equations, therefore, can be conveniently called as “hydrostatic primitive equations”. These equations are simpler in several respects than the complete equation of motion without hydrostatic balance assumption.

One important point to note is that the hydrostatic primitive equations only consider a dry air condition. As cautioned by Simmons and Bengtsson (1984), the formulations should not neglect the local mass of water vapour compared with that of dry air, an approximation that can introduce a small, but not always negligible, error in moist tropical regions. It is thus common for models to use a more accurate form in which the temperature appearing in equations (B.2) and (B.6) is replaced by the virtual temperature, T_v, with a new gas constant for water vapour, R_v, and the specific heat of water vapour at constant pressure, C_{pv}, should be involved in the formulations. Consequently, equation (B.3) for conservation of energy would have to be modified for moisture consideration.

The atmospheric prediction component uses the basic set of equations that describe the time-evolution of the atmosphere. The model integrates forward the hydrostatic primitive equations according to a predicted variable of interest. The basic predicted variables in the primitive equations are in the form of discretised horizontal wind components, u and v, temperature, T, water vapour as represented usually by the specific humidity q, and (in most formulations) surface pressure, p_s. In the Unified Model, the main variables predicted are the zonal and meridional components of the horizontal wind, potential temperature and specific humidity. As an example, a predictive equation for surface pressure p_s is obtained by integrating equation (B.5) from σ = 0 to σ = 1, using the boundary conditions σ = 0 at σ = 0 and σ = 1:

$$\frac{\partial p_s}{\partial t} = -\int_0^1 \nabla \cdot \left(p_s \mathbf{v}\right) \mathrm{d}\sigma \tag{B.7}$$

Vertical velocity is not explicitly predicted, but this variable too can be deduced from equation (B.5), both in the forms of sigma- and pressure-coordinate representing by σ and ω as follows:

$$p_s\dot{\sigma} \;=\; \sigma\int_0^1 \nabla\cdot\left(p_s\mathbf{v}\right)\mathrm{d}\sigma \;-\; \int_0^{\sigma} \nabla\cdot\left(p_s\mathbf{v}\right)\mathrm{d}\sigma \qquad \text{(B.8)}$$

and

$$\omega \;\equiv\; \frac{\mathrm{D}p}{\mathrm{D}t} \;=\; \sigma\mathbf{v}\cdot\nabla p_s - \int_0^{\sigma} \nabla\cdot\left(p_s\,\mathbf{v}\right)\mathrm{d}\sigma \qquad \text{(B.9)}$$

In addition to representing the evolution of the basic atmospheric variables, models generally include various predictive equations for several fields (for example, vorticity and divergence equations). Furthermore, other predicted variables may be introduced depending on a particular application. Currently, ozone and other trace constituents have also been included for many studies. Originally based from the fundamental primitive equations, all those additional equations make the extent of formulations grow in their size, and produce a number of lengthy schemes embedded within a model. All the schemes and their descriptions can be obtained from the literature or more specifically from the relevant model's documentation papers.